Optimal estimation of an observable's expectation value for pure states for general measure of deviation

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We investigate the optimal estimation of quantum expectation value of a physical observable, which minimizes a mean error with respect to general measure of deviation, when a finite number of copies of a pure state are prepared. If pure sates are uniformly distributed, the minimum value of mean error for any measure of deviation is achieved by projective measurement on each copy.

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I. INTRODUCTION

In the quantum information theory, it is one of important issue to pick out the best way how to extract information from given quantum state. Many problems related to this have been investigated for a long time since Helstrom's and Holevo's works [1, 2]. In these problems, many copies of quantum state are needed, because we can get few information from one state, owing to statistical properties of quantum mechanics. The measurements on these copies are classified into two types. One is separable measurement whose positive operator valued measure(POVM) is described by tensor products of POVM on each copy, the other is joint measurements whose POVM includes elements with entangled eigenvectors.

In many cases like quantum state estimation [3, 4, 5, 6, 7], sending some information by qubits [8, 9, 10] and state identification [11, 12], it seems that the joint measurement is optimal. For example, in the paper [11], Hayashi et al. discussed a problem of identifying a given pure state with one of two reference pure states, when no classical knowledge on the reference states is given, but a certain number of copies of them are presented. It was shown that the averaged success probability takes the maximum when we adopt joint measurement on tensor product of reference states and unknown state. However, Masahito Hayashi [13] investigated an estimation with minimum mean error with respect to general measure of deviation which satisfies appropriate conditions and showed that the minimum mean error over all measurement coincides with minimum mean error over separable measurement when the number of copies goes to infinity. For different situations [14, 15, 16], the similar result was obtained.

In the paper [17], D'Ariano et al. discussed the optimal estimation of an observable's expectation value for the finite copies of pure-state and showed that the measurement which minimizes the variance is separable one if POVM is unbiased namely averaged estimator over the repeated measurement is equal to the expectation value. In the previous paper [18], when N copies of pure state distributed uniformly in d-dimensional Hilbert space is presented, it was seen that the measurement which minimizes the averaged variance over all pure state is separable measurement on each copy of N pure states. Thus for the estimation of observable's expectation value, it seems that separable measurement is optimal. In this paper, in order to make sure of this statement, we investigate whether minimum value of mean error with respect to any measure of deviation is achieved by separable measurement of each state or not.

II. ESTIMATION WITH MINIMUM MEAN ERROR FOR GENERAL MEASURE OF DEVIATION

In the previous paper [18], we determined the optimal way to estimate expectation value of a physical observable Ω, namely, POVM and estimator for each element of it which minimize the variance, when N copies of unknown pure state $|\phi\rangle$ on d-dimensional Hilbert space are prepared. In the same situation we find the POVM $\{E_a\}$ and estimator ω_a which minimize the mean error $\langle\Delta\rangle$ defined by,

$$\langle\Delta\rangle = \left\langle \sum_{a=1} \mathrm{Tr}[\rho^{\otimes N} E_a] W(\omega_a - \mathrm{Tr}[\Omega\rho]) \right\rangle, \qquad (1)$$

where ρ is density matrix for pure sate $|\phi\rangle$,

$$\rho = |\phi\rangle\langle\phi|,$$

bracket $\langle\cdots\rangle$ means the average for $|\phi\rangle$ which is distributed uniformly over the d-dimensional Hilbert space and the function $W(x)$ is a general measure of deviation. For simplicity, we assume that this function satisfies the conditions;

(a) $W(x) > 0$ $(x \neq 0)$ and $W(0) = 0$,

(b) $\frac{dW(x)}{dx} > 0$ $(x > 0)$, $\frac{dW(x)}{dx} < 0$ $(x < 0)$

(c) $\frac{d^2W(x)}{dx^2} > 0$

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We will discuss the case where $W(x)$ satisfies weaker conditions in the next section.

Expanding the pure state $|\phi\rangle$,

$$|\phi\rangle = \sum_{n=1}^{d} c_n |n\rangle,$$

in the eigenvectors $|n\rangle$, $(n = 1, 2, \cdots d)$ with eigenvalue λ_n of the observable Ω,

$$\Omega|n\rangle = \lambda_n|n\rangle \ \ (n = 1, 2, \cdots, d),$$

the density matrix $\rho^{\otimes N}$ becomes

$$\begin{aligned} \rho^{\otimes N} &= \sum_{n_1=1}^{d} \cdots \sum_{n_N=1}^{d} \sum_{m_1=1}^{d} \cdots \sum_{m_N=1}^{d} c_{n_i} c^*_{m_i} \cdots c_{n_N} c^*_{m_N} \\ &\quad \times |n_1\rangle\langle m_1| \otimes \cdots \otimes |n_N\rangle\langle m_N|, \\ &= \sum_{\substack{s_1, \ldots, s_d \geq 0 \\ s_1 + \cdots + s_d = N}} \sum_{\substack{s'_1, \ldots, s'_d \geq 0 \\ s'_1 + \cdots + s'_d = N}} \\ &\quad \times c_1^{s_1} \cdots c_d^{s_d} (c_1^*)^{s'_1} \cdots (c_d^*)^{s'_d} \\ &\quad \times \sum_{\substack{\{n_1, \cdots, n_N\} \\ (s_1, \cdots, s_d)}} \sum_{\substack{\{m_1, \cdots, m_N\} \\ (s'_1, \cdots, s'_d)}} \\ &\quad \times |n_1\rangle\langle m_1| \otimes \cdots \otimes |n_N\rangle\langle m_N|. \end{aligned} \tag{2}$$

Here, s_n $(n = 1, 2, \cdots, d)$ is occupation number of the eigenstate $|n\rangle$ $(n = 1, 2, \cdots, d)$ so the notation $\sum_{\substack{s_1, \ldots, s_d \geq 0 \\ s_1 + \cdots + s_d = N}}$ means that the summation is taken over all non-negative integers s_l $(l = 1, 2, \cdots, d)$ which satisfy the condition $s_1 + s_2 + \cdots + s_d = N$ and the notation $\sum_{\substack{\{n_1, \cdots, n_N\} \\ (s_1, \cdots, s_d)}}$ means that the summation is taken over all states with same occupation number s_l $(l = 1, 2, \cdots, d)$.

Substituting this expression for the density matrix $\rho^{\otimes N}$ into the definition(1) of $\langle\Delta\rangle$ we want to minimize, we have

$$\begin{aligned} \langle\Delta\rangle &= \frac{1}{Z} \sum_a \int \prod_{n=1}^{d} dc_n dc_n^* \delta \left(\sum_{n=1}^{d} |c_n|^2 - 1 \right) \\ &\quad \times \sum_{\substack{s_1, \ldots, s_d \geq 0 \\ s_1 + \cdots + s_d = N}} \sum_{\substack{s'_1, \ldots, s'_d \geq 0 \\ s'_1 + \cdots + s'_d = N}} \\ &\quad \times c_1^{s_1} \cdots c_d^{s_d} (c_1^*)^{s'_1} \cdots (c_d^*)^{s'_d} \\ &\quad \times \sum_{\substack{\{n_1, \cdots, n_N\} \\ (s_1, \cdots, s_d)}} \sum_{\substack{\{m_1, \cdots, m_N\} \\ (s'_1, \cdots, s'_d)}} \\ &\quad \times |n_1\rangle\langle m_1| \otimes \cdots \otimes |n_N\rangle\langle m_N| \\ &\quad \times W \left(\omega_a - \sum_{n=1}^{d} \lambda_n |c_n|^2 \right), \end{aligned} \tag{3}$$

where $dc_n dc_n^*$ stands for the integration over real part $c_{n\mathrm{R}}$ and imaginary part $c_{n\mathrm{I}}$ of complex number c_n and Z is normalisation factor,

$$\begin{aligned} Z &= \int \prod_{n=1}^{d} dc_n dc_n^* \delta \left(\sum_{n=1}^{d} |c_n|^2 - 1 \right) \\ &= \int \prod_{n=1}^{d} dc_{n\mathrm{R}} dc_{n\mathrm{I}} \delta \left(\sum_{n=1}^{d} |c_n|^2 - 1 \right) = \frac{1}{2} \frac{2\pi^d}{\Gamma(d)}. \end{aligned}$$

Changing variables of integration c_n to ξ_n and φ_n,

$$c_n = \xi_n e^{i\varphi_n} \ \ (n = 1, 2, \cdots, d),$$

the mean error $\langle\Delta\rangle$ becomes

$$\begin{aligned} \langle\Delta\rangle &= \frac{1}{Z} \sum_a \sum_{\substack{s_1, \ldots, s_d \geq 0 \\ s_1 + \cdots + s_d = N}} \sum_{\substack{s'_1, \ldots, s'_d \geq 0 \\ s'_1 + \cdots + s'_d = N}} \\ &\times \left[\int_0^{2\pi} \left(\prod_{n=1}^{d} d\varphi_n \right) e^{i\{(s_1 - s'_1)\varphi_1 + \cdots + (s_d - s'_d)\varphi_d\}} \right. \\ &\times \sum_{\substack{\{n_1, \cdots, n_N\} \\ (s_1, \cdots, s_d)}} \sum_{\substack{\{m_1, \cdots, m_N\} \\ (s'_1, \cdots, s'_d)}} \int_0^{\infty} \left(\prod_{n=1}^{d} d\xi_n \right) \\ &\times \xi_1^{s_1 + s'_1 + 1} \cdots \xi_d^{s_d + s'_d + 1} \delta \left(\sum_{n=1}^{d} \xi_n^2 - 1 \right) \\ &\times \mathrm{Tr}[|n_1\rangle\langle m_1| \otimes \cdots \otimes |n_N\rangle\langle m_N| E_a] \\ &\times \left. W \left(\omega_a - \sum_{n=1}^{d} \lambda_n \xi_n^2 \right) \right]. \end{aligned}$$

The contribution of the term where s_n is different from s'_n vanishes under the $d\varphi_n$ $(n = 1, 2, \cdots, d)$ integration, and we have

$$\langle\Delta\rangle = \sum_{\substack{s_1, \ldots, s_d \geq 0 \\ s_1 + \cdots + s_d = N}} w_{s_1, \cdots, s_d}(\omega_a) \mathrm{Tr}[\mathcal{M}_{s_1 \cdots s_n} E_a], \tag{4}$$

where the function $w_{s_1, \cdots, s_d}(\omega_a)$ and operator $\mathcal{M}_{s_1 \cdots s_n}$ are defied by

$$\begin{aligned} w_{s_1, \cdots, s_d}(\omega_a) &= \frac{(2\pi)^d}{Z} \int_0^{\infty} \prod_{n=1}^{d} d\xi_n \delta \left(\sum_{n=1}^{d} \xi_n^2 - 1 \right) \\ &\quad \times \prod_{n=1}^{d} \xi_n^{2s_n + 1} W \left(\omega_a - \sum_{n=1}^{d} \lambda_n \xi_n^2 \right) \end{aligned} \tag{5}$$

$$\begin{aligned} \mathcal{M}_{s_1 \cdots s_n} &= \sum_{\substack{\{n_1, \cdots, n_N\} \\ (s_1, \cdots, s_d)}} \sum_{\substack{\{m_1, \cdots, m_N\} \\ (s_1, \cdots, s_d)}}, \\ &\quad \times |n_1\rangle\langle m_1| \otimes \cdots \otimes |n_N\rangle\langle m_N|. \end{aligned} \tag{6}$$

Clearly, the operator $\mathcal{M}_{s_1\cdots s_n}$ is non-negative and its rank is equal to one;

$$\mathcal{M}_{s_1\cdots s_n} = \left(\sum_{\substack{\{n_1,\cdots,n_N\}\\(s_1,\cdots,s_d)}} |n_1\rangle \otimes \cdots \otimes |n_N\rangle \right) \times \left(\sum_{\substack{\{m_1,\cdots,m_N\}\\(s_1,\cdots,s_d)}} \langle m_1| \otimes \cdots \otimes \langle m_N| \right).$$

From the conditions (a) $\sim$ (c) for $W(x)$, we can see that the first derivative of function $w_{s_1,\cdots,s_d}(\omega_a)$ becomes positive(negative) as x goes to $+\infty(-\infty)$ and that the second derivative of it is positive. Thus there is only one value $\Omega^{(\min)}_{s_1,\cdots,s_d}$ where the first derivative of the function $w_{s_1,\cdots,s_d}(\omega_a)$ vanishes and the function $w_{s_1,\cdots,s_d}(\omega_a)$ takes the minimum. As the operator $\mathcal{M}_{s_1\cdots s_n}$ is non-negative, we can get lower limit of mean error $\langle\Delta\rangle$,

$$\begin{aligned}
\langle\Delta\rangle &= \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}} w_{s_1,\cdots,s_d}(\omega_a)\mathrm{Tr}[\mathcal{M}_{s_1\cdots s_d}E_a],\\
&\geq \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}} w_{s_1,\cdots,s_d}(\Omega^{(\min)}_{s_1,\cdots,s_d})\\
&\times\ \mathrm{Tr}[\mathcal{M}_{s_1\cdots s_n}E_a],\\
&= \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}} w_{s_1,\cdots,s_d}(\Omega^{(\min)}_{s_1,\cdots,s_d})\\
&\times\ \mathrm{Tr}\left[\mathcal{M}_{s_1\cdots s_n}\left(\sum_a E_a\right)\right],\\
&= \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}} w_{s_1,\cdots,s_d}(\Omega^{(\min)}_{s_1,\cdots,s_d})\\
&\times\ \mathrm{Tr}\left[\mathcal{M}_{s_1\cdots s_n}\right],
\end{aligned} \tag{7}$$

Now, we prove that this lower limit is achieved if we choose $\mathcal{P}_{s_1\cdots,s_d}$ which is defined by

$$\mathcal{P}_{s_1\cdots,s_d} = \sum_{\substack{\{n_1,\cdots,n_N\}\\(s_1,\cdots,s_d)}} |n_1\rangle\langle n_1| \otimes \cdots \otimes |n_N\rangle\langle n_N|, \tag{8}$$

and which satisfies the condition for projective operators

$$\sum_{\substack{\{n_1,\cdots,n_N\}\\(s_1,\cdots,s_d)}} \mathcal{P}_{s_1\cdots,s_d} = \mathbf{1},$$
$$\mathcal{P}_{s_1\cdots,s_d}\mathcal{P}_{s'_1\cdots,s'_d} = \delta_{s_1 s'_1}\cdots\delta_{s_d s'_d}\mathcal{P}_{s_1\cdots,s_d}.$$

as the POVM and $\Omega^{(\min)}_{s_1,\cdots,s_d}$ as the estimator for each element of this POVM.

Under this choice of POVM and estimators, the mean error $\langle\Delta\rangle$ in equation (4) becomes

$$\begin{aligned}
\langle\Delta\rangle &= \sum_{\substack{s'_1,\ldots,s'_d\geq 0\\ s'_1+\cdots+s'_d=N}} \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}}\\
&\times\ w_{s_1,\cdots,s_d}(\Omega^{(\min)}_{s'_1,\cdots,s'_d})\mathrm{Tr}[\mathcal{M}_{s_1\cdots s_n}\mathcal{P}_{s'_1,\cdots,s'_d}]\\
&= \sum_{\substack{s_1,\ldots,s_d\geq 0\\ s_1+\cdots+s_d=N}}\\
&\times\ w_{s_1,\cdots,s_d}(\Omega^{(\min)}_{s_1,\cdots,s_d})\mathrm{Tr}[\mathcal{M}_{s_1\cdots s_n}],
\end{aligned}$$

since $\mathcal{P}_{s_1\cdots,s_d}$ and $\mathcal{M}_{s_1\cdots s_n}$ satisfy the following relation

$$\begin{aligned}
\mathcal{M}_{s_1\cdots s_n}\mathcal{P}_{s'_1\cdots,s'_d} &= \mathcal{P}_{s'_1\cdots,s'_d}\mathcal{M}_{s_1\cdots s_d}\\
&= \delta_{s_1 s'_1}\cdots\delta_{s_d s'_d}\mathcal{M}_{s_1\cdots s_n}.
\end{aligned}$$

Thus we can obtain the POVM $\{\mathcal{P}_{s_1,\cdots,s_d}\}$ and estimators $\Omega^{(\min)}_{s_1,\cdots,s_d}$ which minimizes the mean error $\langle\Delta\rangle$ for any measure of deviation . Eigenvectors of some elements of this POVM are entangled states of N copies (i.e any linear combination of states with same occupation number) and the measurement for this POVM is joint one. However, in the light of the facts that elements of this POVM are linear combination of the projection operators $|n_1\rangle\langle n_1| \otimes \cdots \otimes |n_N\rangle\langle n_N|$ and that the mean error is linearly dependent on the element of POVM, we have another optimal measurement described by projection operators $P_{n_1\cdots n_N}$ $(n_1,\cdots,n_N = 1,2,\cdots,d)$

$$P_{n_1\cdots n_N} = |n_1\rangle\langle n_1| \otimes \cdots \otimes |n_N\rangle\langle n_N|,$$

and estimator $\Omega^{(\min)}_{s_1,\cdots,s_d}$,where $\{s_1,\cdots,s_d\}$ is occupation number of the state after measurement, namely, the state where the i-th state is the eigenvector $|n_i\rangle$, and this is separable measurement.

III. SUMMARY AND DISCUSSION

In this paper, when N copies of pure state which is distributed uniformly in d-dimensional Hilbert space are prepared, we showed that the separable measurement is optimal in the sense that the mean error with respect to any measure of deviation which satisfies the condition (a) $\sim$ (c) takes the minimum.

The POVM is independent of the choice of measure $W(x)$ of deviation , although the estimator $\Omega^{(\min)}_{s_1,\cdots,s_d}$ is dependent on it. For the case with $d = 2$, $N = 2$ and $\Omega = \sigma_z$, when we choose

$$W(x) = \sigma^2 \sinh^2 \frac{x}{\sigma}, \tag{9}$$

as measure of deviation, the estimator ω_{s_1,s_2} is given by

$$\begin{aligned}
\omega_{2,0} &= \frac{\sigma}{4}\log\left\{\frac{(\sigma^2-2\sigma+4)\sinh\frac{2}{\sigma}-2(\sigma-2)\cosh\frac{2}{\sigma}}{(\sigma^2+2\sigma+4)\sinh\frac{2}{\sigma}-2(\sigma+2)\cosh\frac{2}{\sigma}}\right\}\\
\omega_{1,1} &= 0\\
\omega_{0,2} &= \frac{\sigma}{4}\log\left\{\frac{(\sigma^2+2\sigma+4)\sinh\frac{2}{\sigma}-2(\sigma+2)\cosh\frac{2}{\sigma}}{(\sigma^2-2\sigma+4)\sinh\frac{2}{\sigma}-2(\sigma-2)\cosh\frac{2}{\sigma}}\right\}\\
&= -\omega_{2,0}
\end{aligned}$$

which is dependent on the parameter σ.

The behavior of the estimator $\omega_{2,0}$ in the FIG.1 shows that the limit of $\omega_{2,0}$ as σ approaches infinity is equal to $\frac{1}{2}$ which is obtained from the result in the previous paper [18] as is expected from the fact that the limit of measure of deviation $W(x)$ as σ tends to infinity is equal to x^2.

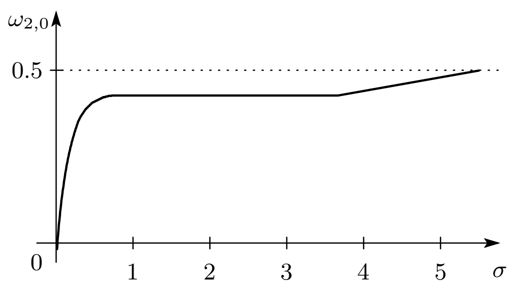

FIG. 1: The estimator $\omega_{2,0}$ as a function of parameter σ in measure(9) of deviation. When σ goes to infinity, the estimator $\omega_{2,0}$ approaches $\frac{1}{2}$ which is equal to the estimator for the case with $d=2$, $N=2$ and the observable σ_z in the previous paper [18].

We assumed that the measure of deviation satisfies the conditions (a) $\sim$ (c). These conditions are sufficient conditions for existence of unique value $\Omega^{(\min)}_{s_1,\cdots,s_d}$ where the function $w_{s_1,\cdots.s_d}(\omega_a)$ defined by the equation (5) takes the minimum. Even if conditions (b) and (c) are replaced with weaker condition (b')

(b') the function $W(x)$ is continuous and monotonically decreasing (increasing) function if x is negative (positive),

existence of minimum value of the function $w_{s_1,\cdots.s_d}(\omega_a)$ is shown as follows. Because the function $w_{s_1,\cdots.s_d}(\omega_a)$ is continuous, in the finite region between the smallest eigenvalue $\lambda_{\min}$ and the largest eigenvalue $\lambda_{\max}$ of observable Ω , there is, at least, one value $\Omega'^{(\min)}_a$ at which the function $w_{s_1,\cdots.s_d}(\omega_a)$ takes the minimum. For the ω_a which is smaller than the smallest eigenvalue $\lambda_{\min}$ of observable Ω, because $\omega_a-\sum_{n=1}^d\lambda_n\xi_n^2$ is negative;

$$\omega_a-\sum_{n=1}^d\lambda_n\xi_n^2\le\omega_a-\lambda_{\min}<0,$$

the measure of deviation $W(\omega_a-\sum_{n=1}^d\lambda_n\xi_n^2)$ is monotonically decreasing function of variable ω_a and the function $w_{s_1,\cdots.s_d}(\omega_a)$ is monotonically decreasing function of variable ω_a, too. Similarly, for the ω_a which is larger than the largest eigenvalue $\lambda_{\max}$, $w_{s_1,\cdots.s_d}(\omega_a)$ is monotonically increasing function of variable ω_a. Hence, at $\omega_a=\Omega'^{(\min)}_a$ this function becomes minimum over the interval $(-\infty,+\infty)$.

In the case with the condition (b') in stead of the conditions (b) and (c) , we may have some points at which the function $w_{s_1,\cdots.s_d}(\omega_a)$ take the minimum and cannot uniquely determine estimator corresponding to each element of projection operator.

We investigated the case where pure state is distributed uniformly. Because of this assumption, integrand in the equation (4) except for the factor $\prod_{i,j}^N c_{n_i}c^*_{m_i}$ is independent of complex argument φ_i of coefficient c_n and we obtained the equation (4). More generally, for the case where pure states are distributed by the probability $p(|c_1|,\cdots,|c_N|)$, the mean error $\langle\Delta'\rangle$ becomes

$$\begin{aligned}
\langle\Delta'\rangle &= \sum_a\int\prod_{n=1}^d dc_n dc_n^*\delta\left(\sum_{n=1}^d|c_n|^2-1\right)\\
&\times\ p(|c_1|,\cdots,|c_N|)\sum_{n_1=1}^d\cdots\sum_{n_N=1}^d\sum_{m_1=1}^d\cdots\sum_{m_N=1}^d\\
&\times\ \prod_{i,j}^N c_{n_i}c^*_{m_i}\mathrm{Tr}[|n_1\rangle\langle m_1|\otimes\cdots\otimes|n_N\rangle\langle m_N|E_a]\\
&\times\ W(\omega_a-\sum_{n=1}^d\lambda_n|c_n|^2),
\end{aligned}$$

and we obtain equation (4) with $w'_{s_1,\cdots,s_d}(\omega_a)$

$$\begin{aligned}
w'_{s_1,\cdots,s_d}(\omega_a) &= (2\pi)^d\int_0^\infty\prod_{n=1}^d d\xi_n\delta\left(\sum_{n=1}^d\xi_n^2-1\right)\\
&\times\ p(\xi_1,\cdots,\xi_N)\prod_{n=1}^d\xi_n^{2s_n+1}\\
&\times\ W\left(\omega_a-\sum_{n=1}^d\lambda_n\xi_n^2\right),
\end{aligned}$$

instead of $w_{s_1,\cdots,s_d}(\omega_a)$. In the same manner as in the previous section, it is shown that the separable measurement on each copy is optimal. For example, we consider measurement of observable σ_z on 2-dimensional Hilbert space. Using polar coordinate (θ,φ) on Bloch sphere, any pure state $|\phi\rangle$ is described in the form,

$$|\phi\rangle=\cos\frac{\theta}{2}|0\rangle+e^{i\varphi}\sin\frac{\theta}{2}|1\rangle,$$

up to phase factor, where $|0\rangle$ and $|1\rangle$ are eigenvectors of σ_z

$$\sigma_z|0\rangle=+1|0\rangle,\ \ \sigma_z|1\rangle=-1|1\rangle.$$

If the probability density for distribution of pure state is independent of azimuthal angle φ, namely, if the distribution of pure state is symmetric under the rotation around

z axis, separable measurement on each copy makes the mean error minimum and is one of the optimal measurements.

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